

On p -degree of elliptic curves

JĘDRZEJ GARNEK

In this note we investigate the p -degree function of an elliptic curve E/\mathbb{Q}_p . The p -degree measures the least complexity of a non-zero p -torsion point on E . We prove some properties of this function and compute it explicitly in some special cases.

1. Introduction

Let $p \neq 2, 3$ be a prime. In this paper we define the **p -degree** of an elliptic curve E over the field \mathbb{Q}_p to be:

$$d_p(E) = \min\{[\mathbb{Q}_p(P) : \mathbb{Q}_p] : P \in E[p], P \neq \mathcal{O}\},$$

where $\mathbb{Q}_p(P)$ denotes the field obtained by adjoining to \mathbb{Q}_p the coordinates of a point $P \in E(\overline{\mathbb{Q}_p})$. It turns out that the p -degree of an elliptic curve with good reduction depends only on the reduction mod p^2 :

Theorem 1.1. *If the elliptic curves $E_1, E_2/\mathbb{Q}_p$ have good reduction and their reductions to \mathbb{Z}/p^2 are isomorphic then $d_p(E_1) = d_p(E_2)$.*

In particular, curves with low p -degree correspond to the canonical lift mod p^2 and in this case we can derive an explicit formula for the p -degree. Let $\text{ord}_p x$ denote the order of x in the group $(\mathbb{Z}/p)^\times$ and $a_p(E)$ be the trace of Frobenius endomorphism for an elliptic curve E defined over a finite field.

Theorem 1.2. *Let E/\mathbb{Q}_p be an elliptic curve with good reduction. Let us consider the following statements:*

- (1) $d_p(E) < p - 1$,
- (2) $E_{\mathbb{F}_p}$ is ordinary and $E_{\mathbb{Z}/p^2}$ is a canonical lift of $E_{\mathbb{F}_p}$,
- (3) $E(\mathbb{Q}_p^{un})[p] \neq 0$, where \mathbb{Q}_p^{un} is the maximal unramified extension of \mathbb{Q}_p ,
- (4) $E_{\mathbb{F}_p}$ is ordinary and $d_p(E) = \text{ord}_p a_p(E)$.

Then (1) implies (2), (2) and (3) are equivalent, and (3) implies (4).

Failures of complementary implications are discussed in Remark 3.3. A result similar to Theorem 1.2 was partially stated already in [3]. Our proof of Theorem 1.2 uses classification of elliptic curves over finite rings by the j -invariant and an effective version of Serre-Tate theorem, which we prove.

Investigating the p -degree is especially interesting when E is a fixed curve over the field of rational numbers \mathbb{Q} and p varies over primes. In particular it is natural to ask about the asymptotic behaviour of the p -degree:

Question 1.3. *Does the p -degree of a fixed elliptic curve tend to infinity as p becomes large?*

The authors of [3] predict that for an elliptic curve without complex multiplication the answer to Question 1.3 is affirmative. They justify this conjecture by a simple heuristics and an averaging result. The heuristics given by David and Weston doesn't work in the case of elliptic curves with CM. We try to verify Question 1.3 in this case. The first ingredient is Theorem 1.2, which yields an explicit formula in the case when p is ordinary. For the remaining primes we apply the following result:

Theorem 1.4. *Let E/\mathbb{Q}_p be an elliptic curve with good supersingular reduction. Then $d_p(E) = p^2 - 1$.*

Combining methods mentioned above we get explicit formulas for the p -degree of elliptic curves with complex multiplication by Gauss and by Eisenstein integers. Let $E_{A,B}$ denote the elliptic curve given by the Weierstrass equation $y^2 = x^3 + Ax + B$.

Theorem 1.5. *Let $D \in \mathbb{Z}$, $D \neq 0$ and assume that $p \nmid 6D$. Then:*

$$d_p(E_{D,0}) = \begin{cases} \text{ord}_p \left((-D)^{\frac{p-1}{4}} \cdot (2s) \right), & \text{for } p \equiv 1 \pmod{4}, \\ p^2 - 1, & \text{for } p \equiv 3 \pmod{4}, \end{cases}$$

$$d_p(E_{0,D}) = \begin{cases} \text{ord}_p \left(-(4D)^{\frac{p-1}{6}} \cdot (2A) \right), & \text{for } p \equiv 1 \pmod{3}, \\ p^2 - 1, & \text{for } p \equiv 2 \pmod{3}, \end{cases}$$

where:

- s is defined for $p \equiv 1 \pmod{4}$ by the equation

$$p = s^2 + t^2$$

and conditions $2 \nmid s$ and $s + t \equiv 1 \pmod{4}$,

- A is defined for $p \equiv 1 \pmod{3}$ by the equation

$$4p = A^2 + 3B^2$$

and condition $A \equiv 1 \pmod{3}$.

Theorem 1.5 and results of Cosgrave and Dilcher from [1] allow us to show that if a certain recurrence sequence contains infinitely many primes, then Question 1.3 has negative answer for elliptic curves with complex multiplication by $\mathbb{Z}[i]$ (cf. Corollary 4.2).

Finally we compute the p -degree for elliptic curves with multiplicative reduction, by reducing the problem to an investigation of the Tate curve (cf. Theorem 4.3).

Outline of the paper. Section 2 provides a quick overview of facts related to elliptic curves over rings, including a classification of elliptic curves over complete rings by their j -invariant and an effective version of Serre-Tate theorem. In Section 3 we prove Theorems 1.1 and 1.2 using the theory of formal groups applied to elliptic curves over finite rings. Finally, in the last section we investigate curves with good supersingular reduction, complex multiplication and bad multiplicative reduction.

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2. Preliminaries

Let R be a commutative unital ring with trivial Picard group (e.g., a local or a finite ring), satisfying $6 \in R^\times$. Any elliptic curve over R (as defined in [6])

is isomorphic to a projective scheme of the form:

$$(2.1) \quad E_{A,B} := \text{Proj}(R[x, y, z]/(y^2z - x^3 - Axz^2 - Bz^3)),$$

for some $A, B \in R$, satisfying: $\Delta(E_{A,B}) := -16 \cdot (4A^3 + 27B^2) \in R^\times$.

Moreover, $E_{A,B} \cong E_{a,b}$ if and only if

$$(2.2) \quad A = u^4 \cdot a, \quad B = u^6 \cdot b \quad \text{for some } u \in R^\times.$$

Note that R -rational points of \mathbb{P}^n are in the correspondence with the points of the "naive" projective space:

$$\mathbb{P}^n(R)_{naive} = \{(x_0, \dots, x_n) \in R^{n+1} : x_0R + \dots + x_nR = R\} / \sim,$$

where $(x_0, \dots, x_n) \sim (y_0, \dots, y_n)$ if and only if $x_i = u \cdot y_i$ for some $u \in R^\times$ and all i . Therefore R -rational points of the Weierstrass curve (2.1) can be identified with elements of $\mathbb{P}^2(R)_{naive}$ that satisfy the Weierstrass equation.

From now on we use the following notation:

- K – a field which is complete with respect to a discrete valuation v ,
- R – the valuation ring of v . It is a local ring with principal maximal ideal $\mathfrak{m} = (\pi)$,
- $k := R/\mathfrak{m}$ – the residue field of v . We assume that it is perfect and of non-zero characteristic $p \neq 2, 3$,
- $R_j := R/\mathfrak{m}^j$ – a local ring with maximal ideal $\mathfrak{m}_j := \mathfrak{m}/\mathfrak{m}^j$. For $j = \infty$ we denote: $R_\infty = R$.

Note that if $i \leq j$, then we can reduce any elliptic curve E/R_j to $E_{R_i} := E \times_{R_j} R_i$ over R_i . In this way we obtain an exact sequence:

$$(2.3) \quad 0 \rightarrow \widehat{E}(\mathfrak{m}_j^i) \rightarrow E(R_j) \rightarrow E_{R_i}(R_i) \rightarrow 0$$

(surjectivity of the reduction follows from Hensel lemma), where \widehat{E} is a one-parameter formal group over R_j . By the general theory of formal groups we have the following isomorphism for $i > \frac{e}{p-1}$:

$$(2.4) \quad \widehat{E}(\mathfrak{m}_j^i) \cong \mathfrak{m}_j^i,$$

that commutes with the reduction. Note that proofs of [9, Theorem IV.6.4.] and [9, Proposition VII.2.2.] remain valid in this case.

It turns out that elliptic curves over R_j are essentially classified by their j -invariant and the isomorphism class of reduction to k . The problem occurs for elliptic curves satisfying $j(E_k) \in \{0, 1728\}$. To treat this issue we introduce the following definition:

Definition 2.1. *The **type** of an elliptic curve $E_{A,B}/R_j$ is (m, n) , if*

$$A = \pi^m \cdot \alpha, \quad B = \pi^n \cdot \beta \quad \text{for } \alpha, \beta \in R_j^\times$$

*and $(\infty, 0)$ (respectively $(0, \infty)$), if $A = 0$ (respectively if $B = 0$).
For a curve $E_{A,B}$ of type $(m, 0)$ (where $1 \leq m < \infty$) we define:*

$$j_1(E_{A,B}) := \frac{B^2}{\alpha^3} \pmod{\mathfrak{m}_j^{j-m}} \in R_{j-m}.$$

Analogously we define $j_1(E_{A,B})$ to be A^3/β^2 for a curve of $(0, n)$ -type.

The condition (2.2) assures that the type of a curve, the j -invariant given by the usual formula and the invariant j_1 do not depend on the choice of the Weierstrass equation.

Lemma 2.2.

- (1) *An isomorphism class of a $(0, 0)$ -type elliptic curve E/R_j is uniquely determined by its reduction E_k and by the j -invariant $j(E) \in R_j$.*
- (2) *Isomorphism class of an elliptic curve E/R_j of type $(\infty, 0)$ or $(0, \infty)$ is determined uniquely by the isomorphism class of E_k .*
- (3) *Assume that $3 \nmid (p-1)$. Then the isomorphism class of a curve E/R_j of type $(m, 0)$ (where $1 \leq m < \infty$) is determined uniquely by the isomorphism class of E_k and $j_1(E) \in R_{j-m}$.*
- (4) *Assume that $4 \nmid (p-1)$. Then the isomorphism class of a curve E/R_j of type $(0, n)$ (where $1 \leq n < \infty$) is determined uniquely by the isomorphism class of E_k and $j_1(E) \in R_{j-n}$.*

Proof. (1) Let us assume that elliptic curves $E_{A,B}/R_j$ and $E_{a,b}/R_j$ satisfy:

$$(2.5) \quad \begin{aligned} E_{A,B} \times_{R_j} k &\cong E_{a,b} \times_{R_j} k, \\ j(E_{A,B}) &= j(E_{a,b}), \end{aligned}$$

where $A, B, a, b \in R_j^\times$. Then by (2.2) there exists $u \in k^\times$ such that

$$A \equiv u^4 \cdot a \pmod{\mathfrak{m}_j}, \quad B \equiv u^6 \cdot b \pmod{\mathfrak{m}_j}.$$

Using (2.5) we obtain:

$$(2.6) \quad A^3 \cdot b^2 = a^3 \cdot B^2.$$

By Hensel lemma the equations:

$$x^4 - A \cdot a^{-1} = 0 \quad \text{and} \quad x^6 - B \cdot b^{-1} = 0$$

have in R_j unique solutions which lift u . Let us denote them by u_1, u_2 respectively. On the other hand the equality (2.6) implies that both u_1 and u_2 satisfy the equation:

$$0 = x^{12} - (A \cdot a^{-1})^3 = x^{12} - (B \cdot b^{-1})^2.$$

Using again Hensel lemma we see that the above equation has a unique solution in R_j . Thus $u_1 = u_2$ and the map $(x, y) \mapsto (u_1^2 \cdot x, u_1^3 \cdot y)$ provides an isomorphism between E_1 and E_2 .

(2), (3), (4) are proven in a similar way. The condition $3|(p-1)$ implies that $\mu_3 \subset k^\times$, which allows us to "twist" a lift of u by a suitable cube root of unity in R_j . Analogously, if $4|(p-1)$ then $\mu_4 \subset k^\times$. \square

Remark 2.3. Note that a curve $E_{0,b}/k$ ($E_{a,0}/k$ respectively) is ordinary if and only if $3|(p-1)$ ($4|(p-1)$ respectively). Other cases will not be relevant for purposes we have in mind.

Let us consider an elliptic curve \mathbb{E} over the residue field k . By a theorem of Serre-Tate ([5, Theorem 1.2.1]) lifts of \mathbb{E} to R_j for $j < \infty$ are determined by lifts of the p -divisible group $\mathbb{E}_k[p^\infty] = (\mathbb{E}_k[p^n])_n$ to R_j . Using the previous classification of elliptic curves we prove an effective version of Serre-Tate theorem:

Theorem 2.4. *Let E_1, E_2 be lifts of an ordinary elliptic curve \mathbb{E}/k to R_j . If $E_1[p^{j-1}]$ and $E_2[p^{j-1}]$ are isomorphic as R_j -group schemes then $E_1 \cong E_2$.*

In order to prove this theorem, we'll have to switch to the algebraic closure \bar{k} . Let $\widehat{K^{un}}$ be the completion of the maximal unramified extension K^{un}/K , with the ring of integral elements A and the maximal ideal $\mathfrak{M} = (\pi)$. We denote $A_j := A/\mathfrak{M}^j$, $\mathfrak{M}_j := \mathfrak{M}/\mathfrak{M}^j$, following our earlier notation.

Lemma 2.5. $(A_j^\times)^{p^m} = (A_j^\times)^{p^{m+1}}$ for $m \geq j - 1$.

Proof. We use induction on j . For $j = 1$ it is immediate. If $a = b^{p^m} \in (A_{j+1}^\times)^{p^m}$ and $m \geq j$, then by induction hypothesis $b^{p^{m-1}} = c^{p^m} + d$ for some $d \in \mathfrak{M}_{j+1}^j$. By raising this equality to the p -th power and using Newton binomial theorem, we get $a = c^{p^{m+1}}$. \square

Proof of Theorem 2.4. By the Serre-Tate theorem lifts of an elliptic curve \mathbb{E}/\bar{k} to A_j are classified by the lifts of its p -divisible group. However, since the residue field of A_j is algebraically closed, the étale-connected sequence (cf. [8, p. 43]) of any p -divisible group G over the ring A_j lifting $\mathbb{E}[p^\infty]$ must be of the form:

$$0 \rightarrow \mu_{p^\infty} \rightarrow G \rightarrow \underline{\mathbb{Q}_p/\mathbb{Z}_p} \rightarrow 0.$$

Thus the lifts of $\mathbb{E}[p^\infty]$ to A_j are classified by:

$$\mathrm{Ext}_{p-\mathbf{div}/A_j}^1(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, \mu_{p^\infty}) = \varprojlim \mathrm{Ext}_{\mathbf{GS}/A_j}^1(\underline{\mathbb{Z}/p^n}, \mu_{p^n}),$$

where $p-\mathbf{div}/A_j$ and \mathbf{GS}/A_j denote respectively categories of p -divisible groups and of group schemes over A_j . The Kummer sequence for flat cohomology (cf. [7, example II.2.18., p. 66]) gives us an isomorphism

$$\mathrm{Ext}_{\mathbf{GS}/A_j}^1(\underline{\mathbb{Z}/p^n}, \mu_{p^n}) \cong H_{fl}^1(A_j, \mu_{p^n}) \cong A_j^\times / (A_j^\times)^{p^n}.$$

Finally, by Lemma 2.5 the natural projection

$$\mathrm{Ext}_{p-\mathbf{div}/A_j}^1(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, \mu_{p^\infty}) \rightarrow \mathrm{Ext}_{\mathbf{GS}/A_j}^1(\underline{\mathbb{Z}/p^{j-1}}, \mu_{p^{j-1}})$$

is an isomorphism. Hence, the lifts of \mathbb{E}/\bar{k} to A_j are classified by the lifts of $\mathbb{E}[p^{j-1}]$. However, Lemma 2.2 assures us that if $E_1, E_2/R_j$ are isomorphic over k and A_j , then they must be also isomorphic over R_j . This ends the proof. \square

3. Proofs of Theorems 1.1 and 1.2

We assume that K/\mathbb{Q}_p is a finite extension and stick to the notation from the previous section. Let also $n = [K : \mathbb{Q}_p]$, $d = [k : \mathbb{F}_p]$ and e be the ramification degree of K/\mathbb{Q}_p . For an abelian group M we define $\mathrm{rank}_p M := \dim_{\mathbb{F}_p} M[p]$. The following lemma computes $\mathrm{rank}_p E_{R_j}(R_j)$ for j big enough and provides a lower bound in the remaining cases. It is a generalization of [3, Lemma 3.1], which computed $\mathrm{rank}_p E_{R_2}(R_2)$ in the unramified case.

Lemma 3.1. *If E/\mathbb{Q}_p has good reduction and $i > \frac{e}{p-1}$ is an integer, then:*

$$\begin{aligned} \text{rank}_p E_{R_j}(R_j) &\geq d \cdot (j - i) + \text{rank}_p E(K), & \text{for } i \leq j < i + e, \\ \text{rank}_p E_{R_j}(R_j) &= n + \text{rank}_p E(K), & \text{for } j \geq i + e. \end{aligned}$$

Proof. Let $E^j := E_{R_j}$ and $E^i := E_{R_i}$. By applying the snake lemma to the sequence (2.3) with multiplication-by- p morphism and using (2.4) we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & E(K)[p] & \longrightarrow & E^i(R_i)[p] \longrightarrow \mathfrak{m}^i/\mathfrak{m}^{i+e} \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathfrak{m}_j^i[p] & \longrightarrow & E^j(R_j)[p] & \longrightarrow & E^i(R_i)[p] \longrightarrow \mathfrak{m}_j^i/\mathfrak{m}_j^{i+e} \end{array}$$

with exact rows. The lower row of the diagram induces an exact sequence:

$$(3.1) \quad 0 \rightarrow \mathfrak{m}_j^i[p] \rightarrow E^j(R_j)[p] \rightarrow \ker(E^i(R_i)[p] \rightarrow \mathfrak{m}_j^i/\mathfrak{m}_j^{i+e}) \rightarrow 0.$$

Elementary computations show that:

$$(3.2) \quad \mathfrak{m}_j^i[p] \cong \begin{cases} (\mathbb{Z}/p)^{d \cdot (j-i)} & \text{for } j < i + e \\ (\mathbb{Z}/p)^n & \text{for } j \geq i + e \end{cases}$$

and that for $j \geq i + e$ the natural projection

$$(3.3) \quad \mathfrak{m}^i/\mathfrak{m}^{i+e} \rightarrow \mathfrak{m}_j^i/\mathfrak{m}_j^{i+e}$$

is an isomorphism. We consider the following two cases:

1) if $i \leq j < i + e$, then by (3.1) and (3.2):

$$\begin{aligned} \text{rank}_p E^j(R_j) &= d \cdot (j - i) + \text{rank}_p \ker(E^i(R_i)[p] \rightarrow \mathfrak{m}_j^i/\mathfrak{m}_j^{i+e}) \\ &\geq d \cdot (j - i) + \text{rank}_p E(K)[p]. \end{aligned}$$

2) if $j \geq i + e$, then by (3.3):

$$\ker(E^i(R_i) \rightarrow \mathfrak{m}_j^i/\mathfrak{m}_j^{i+e}) = \ker(E^i(R_i) \rightarrow \mathfrak{m}^i/\mathfrak{m}^{i+e}) \cong E(K)[p].$$

Thus by (3.1) the sequence

$$0 \rightarrow \mathfrak{m}_j^i[p] \rightarrow E^j(R_j)[p] \rightarrow E(K)[p] \rightarrow 0$$

is exact and the proof follows now from (3.2). \square

Lemma 3.1 plays a central role in proofs of Theorems 1.1 and 1.2. Indeed, the second part of the lemma easily implies Theorem 1.1:

Proof of Theorem 1.1. Let $j = e + \lceil \frac{e}{p-1} \rceil$. Note that $j \leq 2e$ and thus we have a natural homomorphism:

$$\mathbb{Z}/p^2 \rightarrow R_j.$$

Therefore $E_{R_j} = (E_{\mathbb{Z}/p^2})_{R_j}$ and the proof follows from the second part of Lemma 3.1. \square

In order to prove Theorem 1.2 we investigate the p -torsion in fields of low ramification:

Theorem 3.2. *Let K/\mathbb{Q}_p be a finite extension with ramification degree $e < p - 1$. We keep notation introduced in Section 2. For an elliptic curve E/\mathbb{Q}_p with good reduction the following conditions are equivalent:*

- (1) $E(K)[p] \neq 0$,
- (2) $E_k(k)[p] \neq 0$ and $\text{rank}_p E_{R_2}(R_2)[p] = d + 1$,
- (3) $E_k(k)[p] \neq 0$ and $E_{\mathbb{Z}/p^2}$ is the canonical lift of $E_{\mathbb{F}_p}$,
- (4) $E(K \cap \mathbb{Q}_p^{un})[p] \neq 0$.

Proof. (1) \Rightarrow (2). By (2.4) it follows that $E(K)[p] \hookrightarrow E_k(k)[p]$, so that $E_k(k)[p] \neq 0$. Therefore the étale-connected sequence of $E_{R_2}[p]$ is of the form:

$$(3.4) \quad 0 \rightarrow \mu_p \rightarrow E_{R_2}[p] \rightarrow \underline{\mathbb{Z}/p} \rightarrow 0$$

and thus:

$$\text{rank}_p E_{R_2}(R_2) \leq \text{rank}_p \mu_p(R_2) + \text{rank}_p(\underline{\mathbb{Z}/p}) = d + 1.$$

The equality follows from the first part of Lemma 3.1 applied to $(i, j) = (1, 2)$.

(2) \Rightarrow (3). By comparing ranks, we see that the image of $\mu_p(R_2)$ in $E_{R_2}(R_2)[p]$ has index p . Let $g \in E_{R_2}(R_2)[p]$ be an element not belonging to the image of $\mu_p(R_2)$. Then g corresponds to a morphism $\underline{\mathbb{Z}/p} \rightarrow E[p]$, which (after an eventual twist by an automorphism of $\underline{\mathbb{Z}/p}$) provides a section of the étale-connected sequence (3.4). Thus this sequence splits and E_{R_2} is a canonical lift of E_k . This implies that $E_{\mathbb{Z}/p^2}$ is a canonical lift of $E_{\mathbb{F}_p}$.

(3) \Rightarrow (4). Let us replace K by $K \cap \mathbb{Q}_p^{un}$. Since E_{R_2} is a canonical lift of E_k :

$$E_{R_2}[p](R_2) \cong \mu_p(R_2) \oplus \underline{\mathbb{Z}/p}(R_2) \cong (\mathbb{Z}/p)^{d+1}.$$

Thus by Lemma 3.1 $\text{rank}_p E(K) = \text{rank}_p E_{R_2}(R_2) - d = 1$.

The implication (4) \Rightarrow (1) is obvious. \square

Proof of Theorem 1.2. Theorem 3.2 easily implies (1) \Rightarrow (2) \Leftrightarrow (3). The implication (3) \Rightarrow (4) follows from Theorem 3.2 and [3, Lemma 4.3.]. \square

Remark 3.3. The condition (4) of Theorem 1.2 doesn't imply (3) in general. In order to see this consider the curve $E_{1,1}/\mathbb{Q}_5$. Its division polynomial Ψ_5 factors into two polynomials of degrees 2 and 10. Therefore $d_5(E) \in \{2, 4\}$. Suppose, for a contradiction, that $E(\mathbb{Q}_5^{un})[5] \neq 0$. Then Theorem 3.2 implies that $E(K)[5] \neq 0$ for some unramified extension K/\mathbb{Q}_p of degree ≤ 4 . On the other hand the degree 2 factor of Φ_5 is of the form:

$$x^2 + (2 \cdot 5 + 4 \cdot 5^2 + \dots)x + (2 \cdot 5^{-1} + 2 + 4 \cdot 5^2 + 5^3 + \dots).$$

One easily checks that roots of the latter polynomial are ramified. This implies that $E(\mathbb{Q}_5^{un})[5] = 0$ and $d_5(E) = 4$ (since $d_5(E) = 2$ would imply $E(\mathbb{Q}_5^{un})[5] \neq 0$ by Theorem 1.2). Straightforward calculation shows that $\text{ord}_5 a_5(E) = 4$.

The implication (2) \Rightarrow (1) is not true either. Indeed, let E/\mathbb{Q}_5 be the canonical lift of $E_{1,1}/\mathbb{F}_5$. Then E clearly satisfies the condition (2), but $d_5(E) = \text{ord}_5 a_5(E) = 4$. One constructs counterexamples for other primes in a similar way.

The Theorem 1.2 shows that the elliptic curves with low p -degree are well understood. It motivates the following question, which seems to remain open:

Question 3.4. *What values larger than $(p-1)$ can the p -degree attain for a fixed p ? Are there any divisibility conditions on $d_p(E)$, e.g. $d_p(E) \mid (p^2 - 1)$?*

4. Supersingular, CM and multiplicative reduction curves

In the final section we compute the p -degree explicitly in some special cases. We start with elliptic curves with supersingular reduction. Theorem 1.4 will be proven by means of formal groups.

Proof of Theorem 1.4. Note that the multiplication-by- p morphism on \widehat{E} must be of the form:

$$[p](T) = pf(T) + g(T^{p^2}),$$

where $f, g \in R[[T]]$, $g(0) = 0$, $f(T) = T + \dots$ (cf. Proposition IV.2.3. (a), Corollary IV.4.4. and Theorem IV.7.4. from [9]). Thus, if $E(K)[p] \neq 0$ then for some $x \in \mathfrak{m}$, $x \neq 0$:

$$0 = pf(x) + g(x^{p^2}),$$

which gives:

$$e + v(x) = v(pf(x)) = v(-g(x^{p^2})) \geq v(x^{p^2}) = p^2 \cdot v(x).$$

Thus, we get:

$$[K : \mathbb{Q}_p] \geq e = (p^2 - 1) \cdot v(x) \geq p^2 - 1.$$

□

Corollary 4.1. *Let E/\mathbb{Q} be an elliptic curve with complex multiplication by $R_K = \mathbb{Z}[\omega]$, the maximal order in an imaginary quadratic field K . Then for any prime $p \nmid \Delta(E/\mathbb{Q})$:*

$$d_p(E) = \begin{cases} p^2 - 1, & \text{if } p \text{ is inert in } R_K, \\ \text{ord}_p(a_p(E)), & \text{if } p \text{ splits in } R_K. \end{cases}$$

Proof. If p is inert in R_K then by Deuring criterion, E is supersingular at p and the proof follows from Theorem 1.4. In case if p splits in R_K , E is ordinary at p . Moreover, an elliptic curve with complex multiplication is always the canonical lift of its reduction, so it suffices to use Corollary 1.2 to finish the argument. □

Theorem 1.5 follows now easily by applying the explicit trace formulas.

Proof of Theorem 1.5. By [4, Theorem 18.5]:

$$a_p(E_{D,0}) = \overline{(-D/\pi)_4} \cdot \pi + (-D/\pi)_4 \cdot \bar{\pi},$$

where $p = \pi\bar{\pi}$ and $\pi \equiv 1 \pmod{2+2i}$. The formula for $d_p(E_{D,0})$ follows if we observe that $\pi = s + it$ and

$$(-D/\pi)_4 \equiv (-D)^{(p-1)/4} \pmod{\pi}, \quad \bar{\pi} \equiv 2s \pmod{\pi}.$$

Analogously we get the formula for $d_p(E_{0,D})$, by using [4, Theorem 18.4]. □

Applying the formula from Theorem 1.5 we see that Question 1.3 from the Introduction in the special case of $E_{D,0}$ comes down to looking for primes in some recurrence sequences.

Corollary 4.2. *Let $D \in \mathbb{Z}$, $D \neq 0$ and let $p \nmid 2D$.*

- (a) *If $d_p(E_{D,0}) \in \{1, 2, 4\}$ then $p = 5$.*
- (b) *$d_p(E_{D,0}) = 8$ if and only if p is of the form $a_k^2 + a_{k+1}^2$ for some $k \geq 0$, where:*

$$a_0 = 0, \quad a_1 = 1, \quad a_{k+2} = 4a_{k+1} - a_k.$$

Proof. Note that $\text{ord}_p\left((-D)^{\frac{p-1}{4}}\right) \mid 4$. Thus:

- $d_p(E) \in \{1, 2, 4\}$ implies that $\text{ord}_p(2s) \in \{1, 2, 4\}$,
- $d_p(E) = 8$ if and only if $\text{ord}_p(2s) = 8$.

The proof follows now from Theorems 1, 2 and 3 in [1]. □

Finally we investigate the case of curves with multiplicative reduction.

Theorem 4.3. *Let $E = E_{A,B}$ be an elliptic curve over \mathbb{Q}_p with multiplicative reduction. Then:*

$$d_p(E) = \begin{cases} p-1, & \text{if } j(E) \notin \mathbb{Q}_p^p \\ 1, & \text{if } j(E) \in \mathbb{Q}_p^p, \gamma(E/\mathbb{Q}_p) \in \mathbb{Q}_p^2 \\ 2, & \text{if } j(E) \in \mathbb{Q}_p^p, \gamma(E/\mathbb{Q}_p) \notin \mathbb{Q}_p^2, \end{cases}$$

where $\gamma = -2A/B$ and $\mathbb{Q}_p^2, \mathbb{Q}_p^p$ are the sets of squares and p -th powers in \mathbb{Q}_p respectively.

Proof. Let $L = \mathbb{Q}_p(\sqrt{\gamma})$ and $x = 1/j(E)$. Then E/\mathbb{Q}_p is isomorphic over L to the Tate curve E_q , where q is given by the power series:

$$(4.1) \quad g(x) = x + 744x^2 + \dots \in \mathbb{Z}[[x]]$$

(cf. [10, Theorem V.5.3]). In other words, there is an isomorphism of $\text{Gal}(\overline{\mathbb{Q}_p}/L)$ -modules $\Psi : \overline{\mathbb{Q}_p}^\times / \langle q \rangle \rightarrow E(\overline{\mathbb{Q}_p})$ satisfying:

$$(4.2) \quad \Psi \circ \sigma = \chi(\sigma) \cdot \Psi \quad \text{for every } \sigma \in \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p),$$

where $\chi : \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \rightarrow \{\pm 1\}$ denotes the quadratic character associated to L/\mathbb{Q}_p . Observe that for $L = \mathbb{Q}_p$, the character χ is trivial. Let $P = \Psi(z) \in$

$E[p]$, $P \neq \mathcal{O}$, where:

$$z^p = q^j \quad \text{for some } j \in \{0, 1, \dots, p-1\}.$$

Note that for $\sigma \in \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ we have $\sigma \in \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(P))$ if and only if:

$$\Psi(\sigma(z)) = \sigma(\Psi(z)),$$

which is equivalent by (4.2) to an alternative:

$$(4.3) \quad \sigma \in \text{Gal}(\overline{\mathbb{Q}}_p/L(z)) \quad \text{or} \quad \begin{cases} \sigma(z) \equiv z^{-1} \pmod{q^{\mathbb{Z}}} \\ \sigma \notin \text{Gal}(\overline{\mathbb{Q}}_p/L) \end{cases}.$$

However, $\sigma(z) \equiv z^{-1} \pmod{q^{\mathbb{Z}}}$ implies that

$$q^{2j} = (z \cdot \sigma(z))^p \equiv 1 \pmod{q^{p\mathbb{Z}}},$$

which is possible, if and only if, $j = 0$. Thus, if $j \neq 0$, then we must have $\mathbb{Q}_p(P) = L(z)$, by the equality of absolute Galois groups. In order to finish the proof we consider two cases:

Case I: Assume that $j \neq 0$. Note that $\mathbb{Q}_p(z)/\mathbb{Q}_p$ is totally ramified and L/\mathbb{Q}_p is unramified, thus $L \cap \mathbb{Q}_p(z) = \mathbb{Q}_p$ and:

$$\begin{aligned} [\mathbb{Q}_p(P) : \mathbb{Q}_p] &= [L(z) : \mathbb{Q}_p] = [\mathbb{Q}_p(z) : \mathbb{Q}_p] \cdot [L : \mathbb{Q}_p] = \\ &= \begin{cases} 1, & z \in \mathbb{Q}_p, \gamma \in \mathbb{Q}_p^2, \\ 2, & z \in \mathbb{Q}_p, \gamma \notin \mathbb{Q}_p^2, \\ \geq p, & z \notin \mathbb{Q}_p. \end{cases} \end{aligned}$$

The condition $z \in \mathbb{Q}_p$ may hold, if and only if, $q \in \mathbb{Q}_p^p$. Using the equality:

$$\mathbb{Q}_p^p = \{p^{pn} \cdot c : n \in \mathbb{Z}, c \in \mathbb{Z}_p^\times, c^{p-1} \equiv 1 \pmod{p^2}\},$$

which follows by the Hensel lemma, and the formula (4.1), we see that $q \in \mathbb{Q}_p^p$, if and only if, $j(E) \in \mathbb{Q}_p^p$.

Case II: Assume that $j = 0$, without loss of generality $z = \zeta_p$. In this case the alternative (4.3) is easily seen to be equivalent to the condition:

$$\sigma(\sqrt{\gamma} \cdot (\zeta_p - \zeta_p^{-1})) = \sqrt{\gamma} \cdot (\zeta_p - \zeta_p^{-1}).$$

Thus $\mathbb{Q}_p(P) = \mathbb{Q}_p(\sqrt{\gamma} \cdot (\zeta_p - \zeta_p^{-1}))$, which yields that $[\mathbb{Q}_p(P) : \mathbb{Q}_p] = p - 1$. The proof is complete. \square

References

- [1] J. B. Cosgrave and K. Dilcher. *The Gauss-Wilson theorem for quarter-intervals*. Acta Math. Hungar., **142**(1), 199–230 (2014).
- [2] D. A. Cox. *Primes of the form $x^2 + ny^2$* . A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York (1989). Fermat, class field theory and complex multiplication.
- [3] Ch. David and T. Weston. *Local torsion on elliptic curves and the deformation theory of Galois representations*. Math. Res. Lett., **15**(3), 599–611 (2008).
- [4] K. Ireland and M. Rosen. *A classical introduction to modern number theory*. Graduate Texts in Mathematics, Vol. 84, Springer-Verlag, New York (1990), second edition.
- [5] N. M. Katz. *Serre-Tate local moduli*. In: Algebraic surfaces (Orsay, 1976–78), Lecture Notes in Math., Vol. 868, pages 138–202, Springer, Berlin-New York (1981).
- [6] N. M. Katz and B. Mazur. *Arithmetic moduli of elliptic curves*. Annals of Mathematics Studies, Vol. 108, Princeton University Press, Princeton, NJ (1985).
- [7] J. S. Milne. *Étale cohomology*. Princeton Mathematical Series, Vol. 33, Princeton University Press, Princeton, NJ. (1980).
- [8] S. S. Shatz. *Group schemes, formal groups, and p -divisible groups*. In: Arithmetic geometry (Storrs, Conn., 1984), pages 29–78. Springer, New York (1986).
- [9] J. H. Silverman. *The arithmetic of elliptic curves*. Graduate Texts in Mathematics, Vol. 106, Springer, Dordrecht (2009), second edition.
- [10] J. H. Silverman. *Advanced topics in the arithmetic of elliptic curves*. Graduate Texts in Mathematics, Vol. 151., Springer-Verlag, New York (1994).

GRADUATE SCHOOL, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
ADAM MICKIEWICZ UNIVERSITY
UMULTOWSKA 87, 61-614 POZNAŃ, POLAND
E-mail address: jgarnek@amu.edu.pl